Positive Mathematical Programming Approaches – Recent Developments in Literature and Applied Modelling

Thomas Heckelei*, Wolfgang Britz and Yinan Zhang

University of Bonn, Germany

Abstract. This paper reviews and discusses the more recent literature and application of Positive Mathematical Programming in the context of agricultural supply models. Specifically, advances in the empirical foundation of parameter specifications as well as the economic rationalisation of PMP models – both criticized in earlier reviews – are investigated. Moreover, the paper provides an overview on a larger set of models with regular/repeated policy application that apply variants of PMP. Results show that most applications today avoid arbitrary parameter specifications and rely on exogenous information on supply responses to calibrate model parameters. However, only few approaches use multiple observations to estimate parameters, which is likely due to the still considerable technical challenges associated with it. Equally, we found only limited reflection on the behavioral or technological assumptions that could rationalise the PMP model structure while still keeping the model's advantages.

1. Introduction

Positive Mathematical Programming (PMP) is an approach to calibrate (agricultural) programming models by introducing non-linear terms in the objective function such that optimality conditions are satisfied at observed levels of decision variables. Applications of PMP date back almost 25 years (Kasnakoglu and Bauer, 1988), but a more widespread use and subsequent discussion in the literature required a general motivation offered by Howitt (1995). A key factor of success was the ability of PMP-type models to generate solutions with realistic diversification of production activities and smooth supply responses without adding weakly justified constraints to the model formulation. The development of PMP coincided with an increased attractiveness of programming models as their explicit technology representation facilitates interdisciplinary research on agri-environmental interaction connecting economic and bio-physical aspects of agricultural systems. Heckelei and Britz (2005) as well as Henry de Frahan et al. (2007) review the use

---

* Corresponding author: thomas.heckelei@ilr.uni-bonn.de.

of the approach in applications and the methodological development until that time. Paris (2011, p. 340-419) provides a recent PMP introduction in the context of a more general and highly interesting textbook on 'symmetric programming'.

Heckelei and Britz (2005) specifically focus on parameter specification and simulation behavior of PMP models and criticize the weak empirical justification and consequently arbitrary model response implied by many of the early applications. They argue in their conclusions that the future use of PMP-type approaches should avoid the arbitrary specification of parameters and dual values of constraints by incorporating prior information in the calibration step, for example on supply elasticities or land values, or go one step further and estimate parameters of non-linear programming models using multiple observations. The authors acknowledge, however, the considerable methodological challenges related to the latter.

This paper aims at assessing the progress with respect to the empirical foundation of PMP-type approaches since the review by Heckelei and Britz (2005). For this purpose we not only look at journal articles that are likely more advanced, but also review the empirical specification and calibration of larger scale programming models regularly used in agricultural and environmental policy analysis. Apart from the empirical foundation, we also like to add another dimension to the assessment by looking at limits and advances of the economic rationale behind PMP-type methodologies thereby picking up an issue already raised by Heckelei (2002) and Heckelei and Wolff (2003).

Despite the fact that we mostly talk about quantitative methodological issues here, we will ourselves rarely use a formal mathematical exposition. The length limit of this paper precludes a suitable technical treatment of the issues considered. Consequently, we run the danger of a less than desirable level of precision at times. We try to accommodate this as much as possible by trying to refer to the literature in a targeted fashion such that the reader may dig into the technical details with the consultation of the references.

The paper comprises two main sections: next, the literature since 2005 is reviewed focusing on methodological advances and distinguishing the three areas calibration, estimation and rationalisation of PMP-type models. The subsequent section reviews some programming models regularly applied in evaluating policies affecting the agri-environmental system with a focus on how PMP is applied and how its application interacts with the remaining structure of the model. Finally, we summarize and conclude.

2. Recent methodological advances of PMP-type approaches

In the next two subsections we stick to parameter specification issues and distinguish here between the two ways Heckelei and Britz (2005) suggested to improve the empirical base of PMP models: using (1) exogenous information on supply responses and shadow prices of resources in calibrating the models to a base year observation on activity levels and (2) estimating programming model parameters in an econometric sense using multiple observations. The final subsection then briefly addresses issues related to the theoretical rationale of PMP models wondering if advances have been made to improve upon the original ad-hoc introduction and motivation of non-linear parameters to the objective function.

2.1 Calibration

In the early or 'standard' approach of PMP a first phase added constraints to the original linear program (LP) forcing production activities to observed levels. Heckelei and Britz (2005) argue in their conclusion that such a first phase is neither necessary nor advisable as shadow prices of resource constraints are set arbitrarily by this procedure and the calibration can be done directly based on the first order conditions of the non-linear model. Furthermore, they strongly recommend the use of prior information such as estimated supply elasticities to define the non-linear terms of the objective function as they dominate the simulation response of the model to changing economic conditions but no information on this is contained in just one observation.

The first publication to be mentioned in this context since 2005 is Henry de Frahan et al. (2007). They calibrate a large sample of Belgian farm models individually, thereby preserving farm heterogeneity in an agricultural sector analysis. Their PMP approach skips the original first step and directly calibrates the models based on the first order conditions of the non-linear constrained optimization model. The constraints relate to milk and sugar quotas but land is assumed to be tradable between farms moving this input into the variable cost component using farmland rental values from the database. The shadow prices of the quotas are approximated by the gross margin differential between the quota products and the next best alternative crop (Henry de Frahan et al. 2007). Non-linear cost terms in the objective function follow the 'traditional' PMP quadratic functional form in activity levels with only diagonal terms determined by the so-called "average cost function" approach (Heckelei, 2002, p. 11)

A modified PMP calibration approach is introduced by Kanellopoulos et al. (2010) in the context of the newly developed Farm Systems SIMulator (FSSIM). It does use a first step with calibration constraints for technical reasons, but sets the shadow price of land to the observed average gross margin, thereby acknowledging that land is the only binding constraint in the base year and consequently captures all returns to fixed factors. A newly introduced parameter then allows for a continuous shift between linear and non-linear cost function parameters while keeping marginal cost constant at the level satisfying the calibration condition. This parameter is directly related to the supply elasticities of the simulation model. For its specification the authors use exogenously given supply elasticities and, alternatively, by an iterative procedure optimizing the forecasting ability across two time periods. The latter outperformed the «Standard PMP» approach based on an out-of sample test. A unique feature of this PMP application is that FSSIM is formulated in rotations rather than using the typical single production activities.

Mérel and Bucaram (2010) point out that most studies using prior information on supply elasticities – which seems to be the most frequently used calibration approach according to our review of models presented below – perform a 'myopic' calibration, i.e. they ignore the change of resource shadow prices when translating the elasticity information to model parameters, a problem already addressed by Heckelei (2002, p. 12 and p. 57). The special contribution of Mérel and Bucaram (2010) is, however, the very rigorous treatment of the question under what conditions a programming model can be exactly calibrated to exogenous own-price supply elasticities. They derive necessary and sufficient conditions for calibrating models with Leontief and CES production functions both combined with quad-

ratic objective function entries. Merel et al. (2011) extend the approach to a generalized CES formulation and hint at the possibility of incorporating more than one constraint.

It is not clear to us if the two technically impressive contributions by Mérel and Bucaram (2010) and Merel et al. (2011) may be adapted to the case where not only own price, but also cross price elasticities are available. When looking at the non-myopic calibration effort by Britz and Adenäuer (2009), a question of perhaps more practical relevance is what happens if the conditions for exact calibration are not fulfilled. Are there any insights to be gained on how closely the model may be calibrated to the given elasticities? Or similar to the discussions of flexible functional forms in behavioral regression models: what are the requirements for the parameterization of constrained programming models to be able to represent any model consistent behavior up to a certain degree of flexibility? This would certainly be also relevant for the estimation of programming models to which we now turn to.

2.2 Estimation

The first steps to move into the direction of econometric estimation of PMP models or, more generally, programming models were already made by Paris and Howitt (1998) who introduced the Generalized Maximum Entropy (GME) principle to the specification of PMP models. The first use of multiple observations was Britz and Heckelei (2000) and an indication of statistically consistent estimation could be provided by Heckelei and Wolff (2003) when using optimality conditions of programming models as estimating equations. Buysse et al. (2007b) introduce the term «Econometric Mathematical Programming» (EMP) for this and consider it a relevant and likely enlarging field for the analysis of multifunctionality issues in agriculture as new appropriate datasets become available. They do cite Heckelei and Britz (2005), however, who concluded that the estimation of programming model parameters in the context of larger, policy relevant models might be methodologically quite challenging and had not yet been shown.

Buysse et al. (2007a) put EMP to practice analyzing the reform of the Common Market Organization in the EU's sugar sector. They use a sample of 117 Belgian sugar beet producing farms across 9 years to estimate parameters of a cost function quadratic in activity levels by employing GME on the first order conditions of the farm programming models. Allowing some parameters to vary between farms they can add restrictions to calibrate each individual farm model to a reference period. Sugar beet quota rents are estimated simultaneously with the cost function parameters using the prior assumptions that quotas are binding and rents cannot be negative. The authors acknowledge that the assumption of binding quotas considerably simplifies an already complex estimation problem as the algorithm does not need to identify the status of the constraints.

The issue of binding or non-binding constraint is addressed methodologically by Jansson and Heckelei (2009) in the context of a spatial equilibrium trade model. A weighted least squares criterion estimates optimality conditions including complementary slackness conditions. Prices and trade cost are simultaneously recovered incorporating the primal and dual side of the model. A simulation exercise shows that this approach is superior in a mean square error sense to previous two-step algorithms to calibrate spatial trade models. They conclude by suggesting that the general approach could also be applied to PMP type models.

Arfini et al. (2008) perform a cross sectional estimation of a quadratic cost function using 50 sample farms from the Emilia-Romagna region. Unlike other farm level PMP-models, the calibrated programming models not only incorporate this cost function but also separately estimated linear, 'farm level' demand functions for the products, thereby endogenising prices. Calibration of individual farm models is achieved by using specific 'deviations' of costs and prices being interpreted as the 'distance' of the farm to costs and prices at the regional level (page 10). We cannot really decide if an endogenous price formulation makes sense in their empirical setting, but the combination of PMP-calibrated farm programming models with a product demand functions formulation is a useful tool in general.

Paris (2011, p. 397-400) – also referring to a 2005 paper in Italian by Arfini and Donati – focuses on the cost function estimation employed by Arfini et al. (2008) within a PMP chapter of his book on symmetric programming. He stresses that this approach of specifying the total variable cost function is similar to the estimation of a dual cost function in econometric models. The difference is seen in the additional structure assumed ('known') and the often negative degrees of freedom when estimating PMP models. In our view the dissimilarities go further and the connection is not fully worked out as a dual cost function returns the minimum cost at given input prices subject to a technology being able to produce the given output quantities. Even if one interpreted activity levels in PMP models as equivalent to output quantities, then the typical PMP non-linear cost function still misses the consistent connection to variable inputs and corresponding prices as determined by an underlying technology. This lack of rationalisation certainly applies to almost all PMP type applications beyond the papers by Arfini, Paris and co-authors and we devote an extra sub-section to it below.

Jansson and Heckelei (2011) provide probably the most extensive estimation exercise up to this date, estimating 217 regional agricultural programming models with 23 crop production activities for the EU. Here, a Bayesian highest posterior density estimator replaces the typical GME applications in the field allowing for a less computationally demanding and – with respect to the employed prior information – a more transparent estimation of the cost function parameters. Dual values of resource constraints (land, quotas) are estimated using average gross margins as a basis for formulating an imprecise prior reflecting limited knowledge before the estimation. Despite the advantages of the Bayesian estimation compared to GME, the authors still had to assume that resource constraints are binding to render the estimation exercise feasible.

So far, all estimation exercises in the field lack the implementation of statistical inference for the estimated parameters. Buysse et al. (2007a) argue that this is not their prime interest as the models' calibration is the main objective (p. 33). Jansson and Heckelei (2011) similarly state that the empirical content of the parameters is higher compared to previous calibration exercises based on one observation (p. 149). The key issue here is however the highly demanding nature of the computations required for hypothesis tests in the context of these complex models. Conceptually, means exist with bootstrapping GME models or simulating Bayesian posterior distributions (Jansson and Heckelei, 2011b).

Another issue – not really restricted to the estimation of programming models – is the so far seemingly arbitrary decision on what model variables are treated as random. For example, whereas Jansson and Heckelei (2011) treat observed acreages, prices, yields,

and input requirements as random, Arfini et al. (2008) implement the same variables as deterministic. The specification in this regard is not discussed in either of the two and seems to perhaps reflect how much the authors identify themselves more with the econometric (random observations) or programming branch (deterministic observations) of the production economics literature.

Gocht (2009, p. 51ff) combines the estimation of a PMP-type model with the reoccurring problem of estimating input allocations to production activities from farm accountancy data. He argues that a simultaneous use of observed activity levels in an optimisation model and observed aggregate input cost categories offers more information for the estimation of input coefficients compared to previous approaches. The general claim is confirmed by an out-of-sample-test for Belgian farm data. The specification of a calibrated farm group model is an automatic side effect of the, admittedly, challenging estimation exercise.

2.3 Rationalisation of PMP models

The above mentioned calibration or estimation papers leave an important question mostly unanswered: What is the economic or technological rationale behind the non-linear terms in the objective function of the simulation model? This question was raised more generally before by Heckelei (2002, p. 51ff) but was not discussed by Heckelei and Britz (2005).

The answer seems central for a proper use of PMP based models. A key argument for their application instead of econometric ones is the facilitated analysis of agri-environmental interactions by explicitly simulating farm management (use of fertilizer, plant protection, tillage, irrigation, etc.). Under the assumption of a Leontief technology, input use increases linearly with the production activity levels and determines their gross margins. If the non-linear PMP terms in the objective function are seen to relate to input use, for example caused by heterogeneous land quality or rotation effects, then they also imply a discrepancy between average and marginal input application rates (and a deviation from a Leontief technology). An environmental indicator using the average rates (input coefficients) would consequently be inconsistent with the model structure. The same discrepancy holds for the calculation of economic indicators based on the Leontief input coefficients.

Are we able to motivate the non-linear costs in a way which preserves the assumption of a Leontief technology for land and intermediate inputs? Let's assume the relation of (opportunity) costs and activity levels ($\mathbf{x}$) not accounted for either by linear constraints ($\mathbf{Ax} \leq \mathbf{b}$) or the variable cost entries in the objective function ($\mathbf{c}$), can be expressed by a non-linear capacity constraint (Heckelei, 2002, p. 30) as $f(\mathbf{x}) = 0$. The capacity constraint might be understood as a non-linear aggregation of labor and capital requirements of the activities bounded by a fixed labor and capital stock. This interpretation is inviting if the linear objective function covers only the difference between revenue ($\mathbf{r}$) and variable costs and labor/capital are not bounded by the linear constraints. The resulting model is:

$$\max_{\mathbf{x}} \pi = (\mathbf{r} - \mathbf{c})' \mathbf{x}$$
$$s.t. \quad \mathbf{Ax} \leq \mathbf{b} \quad [\boldsymbol{\lambda}]$$
$$f(\mathbf{x}) = 0 \quad [\mu] .$$

Its first order optimality conditions differ from those of the usual PMP approach only by the shadow price μ of the capacity constraint (*L* for Lagrangian):

$$\frac{\partial L}{\partial x_i} = c_i + \mathbf{a}_i'\boldsymbol{\lambda} + \frac{\partial f}{\partial x_i}\mu \geq r_i \quad \perp \quad x_i \geq 0 \quad \text{for all i.}$$

One could equivalently define a function g($\mathbf{x}$) for which holds that

$$\frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i}\mu + f(\mathbf{x})\frac{\partial \mu}{\partial x_i} \quad \text{for all i,}$$

remove the capacity constraint and add g($\mathbf{x}$) to the objective function. Consequently, a non-linear objective function could be rationalised with a capacity constraint. However, at this point we are not sure if there exists a functional form for the capacity constraint such that g($\mathbf{x}$) becomes quadratic as often assumed in PMP. For sure it is possible to simply stick to an explicit constraint formulation and Doole et al. (2011) used this to calibrate total milk production on farm as a quadratic function of herd size.

Leaving the behavioural model of profit maximisation behind and not longer assuming non-linearities in the technology, the mean-variance risk model constitutes a fitting and rather straightforward rationalisation of the quadratic non-linear objective function entries (Heckelei, 2022, p. 41). Cortignani and Severini (2009) and Severini and Cortignani (2011) develop a PMP approach that additionally takes risk aversion behavior into account to evaluate insurance schemes. Petsakos et al. (2011) refrain from extra non-linear costs terms such that remaining quadratic objective function entries represent a covariance matrix of gross margins. The authors apply GME to adjust this matrix to perfectly calibrate the model and interpret the resulting matrix as the true covariance matrix.

3. Review of some PMP based models

This section discusses PMP type models with a focus on those designed for repeated application over a longer time horizon and for which documentation was available. Nevertheless, a fully transparent selection rule could not be applied and the selection likely depends on our subjective and limited overview. For each model, a pre-selected list of attributes were collected as far as possible from papers and websites and afterwards verified by the authors of the models[1] which also added missing information (see table 1).

While focusing on PMP based models, we would like to mention, that according to our literature review, the only larger regularly applied model in the European arena not using PMP appears to be Aropaj (DeCara et al., 2005). Besides Aropaj, the programming models of the McCarl school (Schneider et al., 2007: FASOM; Schneider and Schwab, 2006:

EUFASOM; Havlik et al., 2011: GLOBIOM) are other large-scale systems not relying on PMP. And even in a FASOM inspired model, PMP is now used: PASMA (Schmid and Sinabell, 2007), replaces the quadratic terms by a step-wise linear function, a combination of PMP with Linear Programming and convex combination constraints (McCarl, 1982).

3.1 Model types

The review reveals that PMP based models cover a wide range of approaches which might be roughly categorised by three types. The first group comprises bio-economic farm models, typically being quite rich in the technology description and comprising many different activity variants for producing one output. Application of PMP to these model types is relatively new; FFSIM (Kanellopoulos et al., 2010) is taken here as an example. A possible explanation for the limited use could be that researchers dealing with only a few model instances continue to use traditional approaches to model calibration by changing manually coefficients and employing a rich constraint set.

The second strand of models are price exogenous models for aggregate agents, either farm type groups as in FARMIS (Offermann et al., 2005), FIPIM (Arfini et al., 2003), PRO-MAPA (Júdez et al., 2008) and CAPRI-FARM (Gocht and Britz, 2011) or regional models as in RAUMIS (Cypris, 2000), SILAS-Dyn (Mann et al., 2003), CAPRI-REG (Britz and Witzke, 2008) and DRAM (Helming, 2005). Most farm type groups models use single farm records as the source, in Europe often data from the Farm Accounting Data Network (FADN). Exemptions are FIPIM which adds data from IACS, a geo-referenced data base set up for the control of the direct payments claims of the CAP, and CAPRI-FARM which uses the Farm Structure Survey. Most models integrate crop and livestock activities and seem to comprise both animal feed and crop nutrient requirements. SILAS-Dyn seems to be the only recursive-dynamic model in that group and the only one using a capital stock constraint.

Finally, we have two large scale North-American systems in the third group of models which incorporate price endogeneity for outputs: CRAM (Horner et al., 1992) from Canada and REAP (Johansson et al., 2007) for the U.S.. Both combine by now PMP with a spatial equilibrium setup following Takayama and Judge (1971) to incorporate price feedback directly in the model structure. Alternatively to the Takayama-Judge approach, CAPRI and some models covering Member States such as RAUMIS or SILAS-DYN are linked to market models in a more or less coherent way (cf. Britz, 2008) to allow for price feedback. Compared to price exogenous model applications, the price feedback clearly dampens the allocation response in the overall model chain. DRAM is a national approach where regional models are linked to allow for clearing of the manure market. This feature can be switched on in RAUMIS on demand.

3.2 PMP specification in the models

Most models using regional/national data also seem to be linked to some outlook activity, i.e. are used for ex-ante policy assessment and thus face the question if and how to update their PMP terms to a future point in time whereas models at the farm level generally do not project the model specification into the future. Models with a national or regional

focus such as DRAM, PROMAPA or SWAP incorporate specific features such as manure trade or explicit consideration of irrigation activities (PROMAPA). The latter, as well as the use of different intensity variants (CAPRI, PROMAPA, DRAM) or different crop rotations (REAP and FSSIM) might require additional PMP terms to steer substitution between the variants/crop rotations rendering myopic calibration even more questionable.

Some of models reviewed were designed from the beginning to use PMP, other switched to PMP during their lifetime and might have changed the model structure as a consequence, for example RAUMIS dropped flexibility constraints used in earlier versions. In most of the reviewed models, the constraint set is small and the number of decision variables is large so that the allocation response depends to a large extent on the PMP parameters. In most of the models, explicit consideration of capital and labor is missing such that their allocative impact must be captured by the PMP terms.

The importance of the non-linear terms for simulated changes in crop areas and herd sizes and thus output quantities might even be higher than at first glance or as the model documentations might suggest. Many of the constraints mentioned in the model documentations often directly steer other endogenous variables apart from crop acreages and herd sizes. Animal nutrient requirements and other constraints related to feeding often determine the feed mix, nutrient crop requirements often endogenously drive the fertilizer mix. Labor constraints do not impact directly the allocation if the model structures allows for buying labor, but rather define hired labor use. The same holds for restrictions, for example relating to stable places if the model comprises investment possibilities. However, in some models these restrictions are not symmetric as they are not matched with dis-investments or off-farm labor activities. These observations underline that the allocation response in the second strand of PMP based models rests to a large extent on the PMP parameters.

FIPIM, however in the newest estimated version operating for three European regions (Arfini and Donati, 2011), and CAPRI (Jansson and Heckelei, 2011) are the only models reviewed for which PMP parameters are estimated (at least partially). All remaining (and CAPRI for the case of some (perennial) crops not covered by the estimations and for livestock) seem to rely on exogenous elasticities, and all but two seem to employ a myopic calibration method (Heckelei, 2002: 12) which neglects the effects of changing dual values on the simulation response. Here Merel et al. (2010) develop an easy-to-use correction to at least account for the effect of one major constraint such as land. The two exemptions are a variant of the SWAP model used by Merel and specific sub-modules in REAP where crop rotations are employed in conjunction with a CET transformation. Here, several sets of transformation elasticities are introduced in sensitivity experiments. From these, the set which came closest to the elasticities from another model was chosen. In many cases, the sources of the exogenous elasticities used in the calibration step are not found in scientific papers or model descriptions available.

Related to the question of parameter derivation is the question of how dual values of resource constraints are generated, as they might impact the range of stability. Interestingly, despite the criticism and straightforward alternatives to the biased estimates of duals derived in the so-called first stage of the original PMP formulation that are offered in the literature (Júdez et al., 2001; Heckelei and Wolff, 2003; Heckelei and Britz, 2005), many models still rely on it.

Table 1. Overview on reviewed models

Model	Basic model type	Time	Activities	Feed	Crop nutrients	Policy instruments	PMP terms	Risk	Shadow prices	Land	Investment	Labour
RAUMIS	Aggregate regional programming models,	CS EA	31 crops 16 livestock	Endog feed mix from individual products	N,P,K	Production quotas, direct payments, decoupling, set-aside, stocking rate limits, minimum land use and agro-environmental requirements	M			fixed	continuous re-investment costs	Constraints drive hired labour
FARMIS	Aggregate farm type models	CS EA	27 crops 22 livestock	Endog feed mix from individual products	N,P,K	Production quotas, direct payments, decoupling, modulation, set-aside, stocking rate limits, minimum land use requirements	M			fixed	continuous re-investment costs	Constraints drive hired labour
CAPRI	Aggregate regional and farm type models	CS EA	37 crops (all agr. land use) 16 livestock	Endog feed mix from 8 bulks	N,P,K, min/max mineral	Coupled and decoupled premiums, production quotas, ABC sugar market regime, set-aside, greening instruments of CAP	E (annual crops) M rest	only ABC sugar beet regime	Land/quota rents exogenous, rest 1ST	Land supply curve, imperfect substitution arable and grass lands		
DRAM	Linked aggregate regional models	CS	16 arable/3 fodder crops 8 livestock	dry matter from roughage crops for cattle herd		Production quotas for sugar, milk and starch potatoes, direct payments, decoupling, manure policies	M		1ST	fixed		
FFSIM	Bio-economic farm type models (typical farms)	CS			N,P,K		M	Mean-Variance		fixed		
PROMAPA	Aggregate farm type models	CS	31 crops (of which 5 fodder) 4 livestock	Endog feed mix		Coupled and decoupled premiums, production quotas, set-aside, modulation			rental value of land exogenous, rest 1ST	fixed, Irrigated and non-irrigated		
CRAM	Takayama-Judge type model with explicit production function for agriculture		12 crops 6 livestock							fixed		
FIPIM	Takayama-Judge type model with explicit production function for agriculture	CS	11 crops 3 livestock				E			fixed		
REAP	Takayama-Judge type model with explicit production function for agriculture	CS EA		Pre-defined ratios		fixed and counter-cyclical payments, target prices, loan rates, loan deficiency payments, and domestic, agrienvironmental programs	M			fixed, different soil types in regions		
SWAP			12 crops							fixed		
SILAS-DYN	Aggregate regional programming models	RD EA	37 crops including alpine grazing activities 17 livestock	Feed mix from 55 feed types, summer/winter ration	Min (N,P,K, Mg) and max (N,P)	Swiss direct payments system and ecological requirements, Production quotas,and environmental impacts	M		1ST	fixed	Investment activities (buildings/machinery)	Constraints drive hired labour

Abbreviations: CS: comparative static; RD: recursive dynamic; EA: ex-ante; M: myopic calibration against exogenous elasticities; E: estimated; N: nitrogen; P: phosphate; K: Kalium; 1ST: duals from first stage PMP
Note: empty cells imply missing information

Table 1. Continued

Model	Other restrictions	Intensity variants	Env. indicators	Main data sources	Coverage Resolution	Other features	website, model documentation	Software	Graphical User Interface
RAUMIS		-	N balance	National accounts of agriculture Farm structure survey Calculation data	Germany 300 NUTS 3	national linkage possible e.g. of manure trade	http://www.vti.bund.de/no_cache/en/startseite/institutes/rural-studies/research-areas/policy-impact-assessment/vti-modelling-network/raumis.html	FORTRAN	
FARMIS		-	N balance	German regional statistics and FADN	Germany farm type groups		http://www.vti.bund.de/en/startseite/institutes/rural-studies/research-areas/policy-impact-assessment/vti-modelling-network/farmis.html	GAMS	
CAPRI	Production quotas, Set-aside, Premium entitlement, CAP greening instruments, GHG emissions	2 for crops and animals	N,P,K balances, Ammonia and GHG emissions, LCA energy	Regional statistics and FSS Calculation data External outlooks (AGLINK-COSIMO)	EU27/Norway/Turkey/Western Balkans ~1800 farm types / regional models	Sequential calibration with global market model, spatial down-scaling	www.capri-model.org	GAMS	own development, JAVA
DRAM	Maximum amount of N and P from organic and mineral fertilizer per crop per region	8 for dairy cows, 2 for arable crops	Ammonia emissions, N balance	Dutch FADN Agricultural Census Handbooks	Netherlands 66 regions	Manure trade between regions		GAMS	
FFSIM		many		Own surveys, technical coefficient generator, crop growth models	Europe xx typical farms			GAMS	
PROMAPA		Irrigated and non-irrigated crops		Spanish FADN	Spain 140 farm types (specialisation x size x region)			GAMS FORTRAN	ACCESS
CRAM	Flexibility bounds			Statistics Canada, Agriculture and Agri-Food Canada, for the Medium Term Policy Baseline, USDA, for international data	Canada 29 crop production regions up to 4 market regions				
FIPIM					3 regions in Italy farm groups				
REAP		Yield functions from EPIC, crop rotations		USDA production data ERS cost data	US 45 production regions	CET functions for tillage practise and crop rotations		GAMS	
SWAP	Monthly irrigation water				California 33 regions		http://swap.ucdavis.edu/		
SILAS-DYN	Min 7 % ecological Compensation Areas	4 Intensity-Levels	N,P,K, Mg balances Ammonia and GHG emissions LCA energy	Agricultural Information System Economic Accounts for Agricultural Swiss FADN Database	Switzerland 8 zones and 3 prod. Regions	Interaction with Swiss Market Model and Linkage to Swiss Agriculture Life Cycle Assessment (SALCA)		LPL	

Abbreviations: CS: comparative static; RD: recursive dynamic; EA: ex-ante; M: myopic calibration against exogenous elasticities; E: estimated; N: nitrogen; P: phosphate; K: Kalium; 1ST: duals from first stage PMP
Note: empty cells imply missing information

Unfortunately, the review of these longer standing applied models contributes little (if at all) to the question of how to economically rationalise the introduction of non-linear terms on the objective function. Many models do not explicitly consider capital and labor, suggesting that the PMP terms are related to the two primary factors. That interpretation is explicitly used by Arfini and Donati (2011), but hard to defend for models such as RAUMIS, FARMIS or SILAS-DYN where labor and capital are explicitly handled via constraints (fixed factors) or through the objective function (variable factors).

4. Summary and conclusions

Heckelei and Britz (2005) considered the typical PMP application up to this point to have an insufficient empirical base and suggested to either use exogenous information and/or multiple observations. Our literature review and discussion first looks at papers published since this earlier review to see if things have changed. The use of prior information in calibration such as exogenous elasticities or price data for the dual values or resource constraints has clearly increased. Rising awareness of the problem can also be inferred from the fact that some calibration approaches explicitly evaluate the resulting simulation behavior. A highly technical set of papers discusses the required conditions that need to be fulfilled for programming models to be exactly calibrated to a set of exogenous own price elasticities.

Applications of what the literature termed Econometric Mathematical Programming (EMP), i.e. the estimation of programming model parameters based on multiple observations, are still few (regarding number of papers and independent groups engaged in it). While the use of estimators differs probably for good reasons depending on the available data and prior information, divergence in other core assumptions, for example regarding which data are treated as deterministic or stochastic, might be a sign of the still emergent status of that research field. Researchers still face considerable computational challenges for large-scale applications preventing, for example, to relax the assumption that constraints are binding for all observations. Moreover, full statistical inference on estimated parameters is not beyond the conceptual state yet.

A research gap we consider at least as important relates to the lack of a clear rationale, i.e. a combination of behavioral and technological assumptions for the use of typical PMP from model parameterizations. The only exemption is given by papers relying on a mean-variance risk analysis where the quadratic part of the objective function is rationalised by the covariance matrix of uncertain returns. A previously discussed and recently employed alternative of non-linear (capacity) constraints is shown to be completely equivalent to the non-linear objective function entries as long as certain functional restrictions are satisfied. Other behavioral or technological assumptions which would completely move away from typical PMP formulation but still allow for a differentiated analysis of factors affecting agri-environmental interactions could not be identified.

The second part of our review deals with PMP based programming models build for repeated use in policy evaluation exercises. Even though most of these rather diverse models (from small scale bio-economic to national or international scope) are European models, also large-scale, price endogenous, North American models use PMP in different variants. Apparently, the share of (N)LP based models drawing on other calibration

methods has considerably decreased. The models maintained by groups which are also involved in EMP development are often at least in part parameterized by econometric estimations. Most other applications calibrate their models against exogenous elasticities. Early approaches criticised as leading to arbitrary allocation response such as the so-called «standard approach» are abandoned. However, calibration is still mainly done in a myopic way ignoring feedbacks with resource constraints and despite the fact that easy alternatives exist. Equally, many models still use the original first phase of PMP leading to biased shadow prices of binding resources.

The recent developments in the PMP literature clearly move towards a better understanding and improvement of related model specifications. A more solid empirical foundation of models regularly applied in evaluations of agricultural policies or those affecting agriculture can be identified and clearly support an increased reliability of the results. Further improvements in coverage and quality of empirical approaches are still desirable, but the still limited economic rationalisation of PMP-type approaches remains an at least important deficiency. Progress in this area is needed to increase not only scientific acceptability, but also the trust in and understanding of this modeling approach in the policy process. This seems rather important given the increasing relevance of national and global issues requiring sound economic models with technology rich specifications of farm and aggregate agricultural systems.

Acknowledgements

References

Arfini, F. and Donati, M. (2011). *Impact of the Health Check on Structural Change and Farm Efficiency: A Comparative Assessment of Three European Agricultural Regions.* Proceedings of the OECD Workshop on Disaggregated Impacts of CAP Reforms. Paris: OECD, 75-90.

Arfini, F., Donati, M. and Paris, Q. (2003). *A National PMP Model for Policy Evaluation in Agriculture using Micro Data and Administrative Information.* Paper presented at the International Conference Agricultural Policy Reform and the WTO: Where Are We Heading? Capri, Italy, June 23-26, 2003.

Arfini, F., Donati, M., Grossi L. and Paris, Q. (2008). *Revenue and Cost Functions in PMP: a Methodological Integration for a Territorial Analysis of CAP.* Paper presented at the 107th EAAE Seminar Modelling of Agricultural and Rural Development Policies. Sevilla, Spain, January 29 – February 1st, 2008.

Britz, W. (2008). Automated Model Linkages: the Example of CAPRI. *German Journal of Agricultural Economics* 57(8): 363-367.

Britz, W. and Adenäuer, M. (2009). Calibration of Regional Programming Models to Exogenous Supply Elasticities by a Variable Cost Function: The CAPRI-EXPAMOD Example. In: *Integrated Assessment of Agriculture and Sustainable Development (AgSAP)*, March 10-12, 2009, Egmont an Zee (The Netherlands), 52-53.

Britz, W. and Witzke, P. (2008). CAPRI Model Documentation 2008: Version 2. <http://www.capri-model.org/docs/capri_documentation.pdf>, last accessed March 2, 2012.

Buysse, J., Fernagut, B., Harmignie, O., Henry de Frahan, B., Lauwers, L., Polomé, P., Van Huylenbroeck, G. and Van Meensel, J. (2007a). Farm-Based Modelling of the EU Sugar Reform: Impact on Belgian Sugar Beet Suppliers. *European Review of Agricultural Economics* 34(1): 21-52.

Buysse, J., Van Huylenbroeck, G. and Lauwers, L. (2007b). Normative, Positive and Econometric Mathematical Programming as Tools for Incorporation of Multifunctionality in Agricultural Policy Modelling. *Agriculture, Ecosystems and Environment* 120: 70-81.

Cortignani, R. and Severini, S. (2009). Modeling Farm-Level Adoption of Deficit Irrigation using Positive Mathematical Programming. *Agricultural Water Management* 96: 1785-1791.

Cypris, C. (2000). *Positive Mathematische Programmierung (PMP) im Agrarsektormodell RAUMIS*. PhD dissertation, Department of Economic and Agricultural Policy, University of Bonn. Bonn, Germany.

De Cara, S., Houzé, M. and Jayet, P. A. (2005). Green House Gas Emissions from Agriculture in the EU: a Spatial Assessment of Sources and Abatement Costs. *Environmental and Resource Economics* 32(4): 551–583.

Doole, G.D., Ramilan T. and Pannell D. (2011). Framework for Evaluating Management Interventions for Water-Quality Improvement across Multiple Agents. *Environmental Modelling and Software* 26: 860-872.

Gocht, A. (2009). *Methods in Economic Farm Modelling*. PhD dissertation, Department of Economic and Agricultural Policy, University of Bonn. Bonn, Germany.

Gocht, A. and Britz, W. (2011). EU-Wide Farm Type Supply Models in CAPRI - How to Consistently Disaggregate Sector Models into Farm Type Models. *Journal of Policy Modeling* 33(1): 146-167.

Havlík, P., Schneider, A.U., Schmid, E., Böttcher, H., Fritz, S., Skalský, R., Aoki, K., de Cara, S., Kindermann, G., Kraxner, F., Leduc, S., McCallum, I., Mosnier, A, Sauer, T. and Obersteiner, M. (2011). Global Land-Use Implications of First and Second Generation Biofuel Targets. *Energy Policy* 39: 5690-5702.

Heckelei T. and Wolff, H. (2003). Estimation of Constrained Optimisation Models for Agricultural Supply Analysis based on Generalised Maximum Entropy. *European Review of Agricultural Economics* 30(1): 27-50.

Heckelei, T. and Britz, W. (2005). Models based on Positive Mathematical Programming: State of the Art and Further Extensions. In Arfini, F. (eds.), *Modelling Agricultural Policies: State of the Art and New Challenges. Proceedings of the 89th European seminar of the European Association of Agricultural Economics*. Parma, Italy: University of Parma, 48–73.

Heckelei, T. (2002). *Calibration and Estimation of Programming Models for Agricultural Supply Analysis*. Habilitation Thesis, University of Bonn. Bonn, Germany.

Heckelei, T. and Britz, W. (2000). Positive Mathematical Programming with Multiple Data Points: a Cross-Sectional Estimation Procedure. *Cahiers d'Economie et Sociologie Rurales* 57(4): 28-50.

Helming, J.F.M. (2005). *A Model of Dutch Agriculture Based on Positive Mathematical Programming with Regional and Environmental Applications*. PhD dissertation, Wageningen University. Wageningen, Netherlands.

Henry de Frahan, B., Buysse, J., Polomé, P., Fernagut, B., Harmignie, O., Lauwers, L., Van Huylenboreck, G. and Van Meensel, J. (2007). Positive Mathematical Programming for Agricultural and Environmental Policy Analysis: Review and Practice. In: Weintraub, A., Romero, C., Bjørndal, T., Epstein, R. and Miranda, J. (eds.), *Handbook of Operations Research in Natural Resources*. International Series in Operations Research & Management Science, 2007, Volume 99, 1, 129-154.

Horner, G., Corman, J., Howitt, R. E., Carter, C. A. and MacGregor, R. (1992). *The Canadian Regional Agriculture Model, Structure, Operation and Development*. Technical Report 1/92, Ottawa, Ontario, Canada: Agriculture and Agri-Food Canada.

Howitt, R.E. (1995). Positive Mathematical Programming. *American Journal of Agricultural Economics* 77(2): 329-342.

Jansson, T. and Heckelei, T. (2009). A New Estimator for Trade Costs and its Small Sample Properties. *Economic Modelling* 26: 489-498.

Jansson, T. and Heckelei, T. (2010). Estimation of Parameters of Constrained Optimization Models. In: Gilbert, J. (eds.), *Frontiers of Economics and Globalization*, Emerald Group Publishing Limited, Vol. 7, 1-26.

Jansson, T. and Heckelei, T. (2011). Estimating a Primal Model of Regional Crop Supply in the European Union. *Journal of Agricultural Economics* 62(1): 137-152.

Johansson, R., Peters, M. and House, R. (2007). *Regional Environment and Agriculture Programming Model*. Technical Bulletin 1916. United States Department of Agriculture, Economic Research Service.

Júdez, L., Andres, R., Ibanez M., de Miguel, J.M. and Miguel, J.L. (2008). *Impact of the CAP Reform on the Spanish Agricultural Sector*. Paper presented at the 109th EAAE Seminar the CAP after the Fischler Reform: National Implementations, Impact Assessment and the Agenda for Future Reforms. Viterbo, Italy, November 20-21st, 2008.

Júdez, L., Chaya, C., Martinez, S., and Gonsalez, A.A. (2001). Effects of the Measures Envisaged in "Agenda 2000" on Arable Crop Producers and Beef and Veal Producers: an Application of Positive Mathematical Programming to Representative Farms of a Spanish Region. *Agricultural Systems* 67: 121-138.

Kanellopoulos, A., Berentsen, P., Heckelei, T., Van Ittersum, M. and Oude Lansink, A. (2010). Assessing the Forecasting Performance of a Generic Bio-Economic Farm Model Calibrated with Two Different PMP Variants. *Journal of Agricultural Economics* 61(2): 274-294.

Kasnakoglu, H. and Bauer, S. (1988). Concept and Application of an Agricultural Sector Model for Policy Analysis in Turkey. In: Bauer, S. and Henrichsmeyer, W. (eds.), *Agricultural Sector Modelling*, Proceedings of the 16th Symposium of the EAAE, Kiel: Wissenschaftsverlag Vauk, 71-84.

Mann, S., Mack, G. and Ferjani, A. (2003). Können Produktionsentscheidungen als Investitionsentscheidungen modelliert werden? *German Journal of Agricultural Economics* 52(7): 333-340.

McCarl, B.A. (1982). Cropping Activities in Agricultural Sector Models: a Methodological Approach. *American Journal of Agricultural Economics* 64: 768-772.

Merel, P.R. and Bucaram, S. (2010). Exact Calibration of Programming Models of Agricultural Supply against Exogenous Sets of Supply Elasticities. *European Review of Agricultural Economics* 37: 395-418.

Mérel, P.R., Simon, L., and Yi, F. (2011). A Fully Calibrated Generalized Constant-Elasticity-of-Substitution Programming Model of Agricultural Supply. *American Journal of Agricultural Economics* 93(4): 936–948.

Offermann, F., Kleinhanß, W., Hüttel, S. and Küpker, B. (2005). Assessing the 2003 CAP Reform Impacts on German Agriculture using the Farm Group Model FARMIS. In: Arfini, F. (eds.), *Modelling Agricultural Policies: State of the Art and New Challenges. Proceedings of the 89th European seminar of the European Association of Agricultural Economics.* Parma, Italy: Monte Università Parma, 546-564.

Paris, Q. and Howitt, R.E. (1998). An Analysis of ill-Posed Production Problems using Maximum Entropy. *American Journal of Agricultural Economics* 80(1): 124-138.

Paris, Q. (2010). *Economic Foundations of Symmetric Programming.* Cambridge University Press, Cambridge.

Petsakos, A. and Rozakis, S. (2011). *Integrating Risk and Uncertainty in PMP Models.* Paper presented at the EAAE 2011 Congress Change and Uncertainty Challenges for Agriculture, Food and Natural Resources. ETH Zurich, Zurich, Switzerland, August 30 - September 2nd, 2011.

Schmid, E. and Sinabell, F. (2007). On The Choice of Farm Management Practices after the Reform of the Common Agricultural Policy in 2003. *Journal of Environmental Management* 82(9): 332-340.

Schneider, U.A. and Schwab, D.E. (2006). The European Forest and Agricultural Sector Optimization model. Working Paper, Hamburg University, <http://www.fnu.zmaw.de/fileadmin/fnu-files/projects/fasom/EUFASOM_Insea.pdf>, last accessed March 2, 2012.

Schneider, U.A., McCarl, B.A. and Schmid, E. (2007). Agricultural Sector Analysis on Greenhouse Gas Mitigation in US Agriculture and Forestry. *Agricultural Systems* 94: 128–140.

Severini, S. and Cortignani R. (2011). *Modeling Farmer Participation to a Revenue Insurance Scheme by means of Positive Mathematical Programming.* Paper presented at the EAAE 2011 Congress Change and Uncertainty Challenges for Agriculture, Food and Natural Resources. ETH Zurich, Zurich, Switzerland, August 30 – September 2nd, 2011.

Takayama, T. and Judge, G.G. (1971). Spatial and Temporal Price and Allocation Models. *Journal of International Economics* 3(3): 304-304.